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Toward Finite Quantum Field Theories

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The properties that make the $N=4$ super Yang-Mills theory free from ultraviolet divergences are (i) a universal coupling for gauge and matter interactions, (ii) anomaly-free representations, (iii) no charge renormalization, and (iv) if masses are explicitly introduced into the theory, then these are required to satisfy the mass-squared supertrace sum rule $\sum_{s=0,1/2}(-1)^{2s+1}(2s+1)M_s^2=0$. Finite $N=2$ theories are found to satisfy the above criteria. The missing member in this class of field theories are finite field theories consisting of $N=1$ superfields. These theories are discussed in the light of the above finiteness properties. In particular, the representations of all simple classical groups satisfying the anomaly-free and no-charge renormalization conditions for finite $N=1$ field theories are discussed. A consequence of these restrictions on the allowed representations is that an $N=1$ finite $SU(5)$-based model of strong and electroweak interactions can contain at most five conventional families of quarks and leptons, a constraint almost compatible with the one deduced from cosmological arguments.

1. INTRODUCTION

An unattractive feature of field theoretic formulations of quantum electrodynamics and its synthesis with the strong and weak interactions is the process of renormalization of masses and coupling constants by infinite amounts. Although highly successful, it is an additional prescription for handling the theory and is an indication of some feature of fundamental importance lacking in the theory. A new feature of quantum field theories is the existence of monopoles with masses of the order of Planck mass (10^{19} GeV). It was hoped that the quantization condition satisfied by the gauge and the monopole charges could conspire to eliminate the process of renormalization. However such a link has not been forthcoming. Instead, a solution to the problem of ultraviolet divergences has emerged from applying supersymmetry realized in its extended form. This fact has revived interest in supersymmetry once again.

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The $N=4$ super Yang-Mills theory was the first to be proved ultraviolet finite to all orders in perturbation theory (Mandelstam, 1983; Brink et al., 1983a, b; for an alternative proof see Howe et al., 1983, 1984). The proof was greatly facilitated by the formulation of the theory in the light cone gauge. Using the same formalism it has been established that the theory is also finite if mass terms for the scalars and fermions are explicitly added by hand (Taylor, 1983; Namazie et al., 1983; Rajpoot et al., 1983a) provided they satisfy the mass-squared supertrace sum rule

$$\sum_{s=0,1/2} (-1)^{2s+1}(2s+1)M_s^2=0 \tag{1}$$

and also that mass-dependent compensating cubic terms for the scalar bosons are inserted in the theory to ensure full Lorentz invariance of the resulting interactions. Detailed study of the explicitly broken $N=4$ super Yang-Mills theory has revealed that neither supersymmetry nor the internal $SU(4)$ supercharge symmetry of the theory survives. The potential of the theory is bounded from below and can break the gauge symmetry spontaneously (Rajpoot and Taylor, 1983). After spontaneous symmetry breaking the resulting mass spectrum of the particles satisfies the full supertrace sum rule

$$\sum_{s=0,1/2,1} (-1)^{2s+1}(2s+1)M_s^2=0 \tag{2}$$

It has been demonstrated further that these mass-squared sum rules arise from the cancellation of quadratic divergences in the one loop effective potential of the theory (Capper and Rajpoot, 1984). A less satisfying aspect of the $N=4$ theory is that the particle content of the theory, which is $\{1^1, \frac{1}{2}^4, 0^6\}$, where s^d denotes an $SU(4)$ d-dimensional multiplet of particles of spin s, belongs to the adjoint representation of the internal gauge symmetry group G. This is contrary to the quark and lepton representations required to describe the observed spectrum of particles in nature. A question that comes to mind is whether the $N=4$ Yang-Mills theory is the only finite theory or are there others? By studying the anatomy of the $N=4$ theory in the presence of explicit mass terms, we found it has the following properties for finiteness (Rajpoot et al., 1983b): (i) a single coupling constant for gauge and matter interactions; (ii) the representations to which the particles belong are anomaly free; (iii) the coupling constant renormalization vanishes, i.e., $\beta(g)=0$; (iv) the mass term for the scalars and fermions introduced explicitly by hand are required to satisfy the supertrace mass-squared sum rule $\sum_{s=0,1/2}(-1)^{2s+1}(2s+1)M_s^2=0$ before spontaneous symmetry breaking. Note that although these finiteness properties emerge from the study of the explicitly broken $N=4$ theory they are general and do not refer to any specific part of the $N=4$ theory. *On the basis of these properties* *it was conjectured that field theories constructed out of $N = 1$ super multiplets and satisfying the above criteria would also be finite.* In particular the condition for the charge renormalization to vanish at the one loop level in these $N = 1$ theories was derived in Rajpoot et al. (1983b). In passing we note that finite $N = 2$ theories automatically satisfy the above three criteria for finiteness by construction. To see this recall that the particle content of finite $N = 2$ theories is one vector multiplet $\{1^1, \frac{1}{2}^2, 0^2\}$ in the adjoint representation and a finite number of matter hypermultiplets H_i [where now $SU(2)$ is the internal symmetry of the supercharges]. Each hypermultiplet H_i consists of two $N = 1$ matter multiplets $\{\frac{1}{2}^1, 0^2\}$, one in the representation R_i and the other in the conjugate representation $\bar{R}_i$ of the gauge group G. The anomalies of these representations are equal in magnitude but opposite in sign. Since anomalies are additive, this ensures that the hypermultiplet is anomaly free. The number of such hypermultiplets is restricted by the condition of the vanishing of the charge renormalization at one loop (Howe et al., 1983a, b; Koh and Rajpoot, 1984)

$$C_2(G) = \sum_i T(R_i) = \tfrac{1}{2} \sum_i T(H_i) \tag{3}$$

Representations of all classical groups satisfying this condition have been given in Koh and Rajpoot (1984). Note that finite $N = 2$ and $N = 4$ theories have one universal coupling for both the gauge and matter multiplets.

In constructing finite $N = 1$ field theories we use the superfield notation to write down the most general action in terms of $N = 1$ superfields. From this the Lagrangian in components is derived. The main advantage of the superfield approach is to limit the number of arbitrary parameters and interactions. This follows from the fact that the renormalizability of the theory restricts the interaction terms involving superfields to be at most *cubic.* Also, in the presence of abelian gauge symmetry the gauge coupling grows asymptotically and hence the third finiteness condition [i.e., $\beta(g) = 0$] cannot be implemented. Hence only non-Abelian gauge symmetry is admissible. This in turn forbids interaction terms linear in $N = 1$ superfields in the action. With these restrictions in mind the $N = 1$ superfield action is constructed in the following section.

2. VANISHING OF CHARGE RENORMALIZATION

The action with one vector superfield (V) and an arbitrary number of chiral superfields (ϕ^L) in representations R of the internal symmetry gauge group G is

$$S = \int d^4x \left[d^4\theta \, \bar{\phi}_L e^{gV} \phi^L + \frac{1}{64g^2} d^2\theta \, \mathrm{Tr}(W^\alpha W_\alpha) \right.$$

$$
\begin{aligned}
&-d^2\theta\frac{1}{3!}h_{LMN}\phi^L\phi^M\phi^N - d^2\theta\frac{1}{2!}M_{LM}\phi^L\phi^M \\
&-d^2\theta\frac{1}{2!}\bar{M}_{LM}\theta^2\phi^L\phi^M - d^2\theta\, M_\lambda\theta^2\frac{1}{64g^2}\operatorname{Tr}(W^\alpha W_\alpha) \\
&\left.-d^2\theta\, M_\lambda\theta^2 h_{LMN}\phi^L\phi^M\phi^N - d^4\theta\, M_\lambda^2\theta^2\bar{\theta}^2\phi^L\phi_L\right] \\
&+(\text{h.c.})+(\text{gauge fixing})+(\text{compensating ghost})\text{ terms} \qquad (4)
\end{aligned}
$$

where $W_\alpha = \bar{D}^2(e^{-gV}D_\alpha e^{gV})\cdot 1$, $V=V^aT_a$. T_a are the generators of G, $\bar{M}_{LM}\theta^2$ is the spurion that lifts the degeneracy between the masses of the scalars and pseudoscalars; $M_\lambda\theta^2$ is the spurion that gives masses to the gauge fermions (gauginos) of the vector superfield V and repeated indices are summed over. The cubic interaction, which is a group invariant, can be written in different equivalent forms:

$$h_{LMN}\phi^L\phi^M\phi^N = h_{LMN}^{lmn}\phi_l^L\phi_m^M\phi_n^N = h_{L,M,N}^{al,bm,cn}R_{ll'}^a R_{mm'}^{\prime b} R_{nn'}^{\prime\prime c}\phi_{l'}^L\phi_{m'}^M\phi_{n'}^N$$

where R, R', R'' are distinct representations of G. In what follows the symbol R will be taken to represent these distinct representations. In components and after elimination of the auxiliary fields, the Lagrangian is

$$
\begin{aligned}
L = &-\frac{1}{4}G_{\mu\nu}^a G_{\mu\nu}^a + \frac{1}{2}\bar{\lambda}_i^a \not{D}\lambda^a + \frac{1}{2}\bar{\psi}i^L\not{D}\psi^L \\
&+\frac{1}{2}(D^\mu A_L)^2 + \tfrac{1}{2}(D^\mu B_L)^2 - \frac{1}{2}M_{A_L}^2A_L^2 - \frac{1}{2}M_{B_L}^2B_L^2 \\
&-\frac{1}{2}M_L\bar{\psi}_L\psi_L - \frac{1}{2}M_\lambda\bar{\lambda}^a\lambda^a - ig\bar{\lambda}^a(R_{ij}^a)[A_i^L + \gamma_5 B_i^L]\psi_j^L \\
&-\frac{g^2}{2}R_{ij}^aR_{lm}^aA_iA_lB_jB_m - \frac{1}{3!}gh_{LMN}\bar{\psi}^L(A^M+\gamma_5B^M)\psi^N \\
&-M_LA_L(A_L^2+B_L^2) - M_\lambda h_{LMN}(A_LA_MA_N - 3A_LB_MB_N) \\
&-4h_{LMN}h_{LM'N'}A_MB_NA_{M'}B_{N'} + \text{h.c.} \\
&+(\text{gauge fixing})+(\text{ghost})\text{ terms} \qquad (5)
\end{aligned}
$$

(A_L, B_L, λ_L) are the scalar, pseudoscalar, and fermions of ϕ_L transforming according to representation R of G, λ^a are the gauginos and

$$
\begin{aligned}
G_{\mu\nu}^a &= \partial_\mu W_\mu^a - \partial_\nu W_\mu^a + gf^{abc}W_\mu^b W_\nu^c \\
D^\mu A^L &= \partial^\mu A^L - ig(W^\mu\cdot R)A_L \\
\not{D} &= \partial_\mu D_\mu
\end{aligned}
$$

The mass matrices have been taken to be diagonal for simplicity; $M_{LM} = M_L\,\delta_{LM}$, $\bar{M}_{LM} = \bar{M}_L\,\delta_{LM}$, $M^2_{A_L} = M^2_L + \bar{M}^2_L + M^2_\lambda$, $M^2_{B_L} = M^2_L - \bar{M}^2_L + M^2_\lambda$. Gauge invariance requires the Yukawa coupling and the mass matrices to satisfy the following relations:

$$h_{XMN}\tilde{R}^{aX}_L C^{-1} + h_{LYN}C^{-1}R^{aY}_M + h_{LMZ}C^{-1}R^{aZ}_N = 0 \tag{6}$$

$$\tilde{R}^a C^{-1} M_L + C^{-1} M_L R^a = 0 \tag{7}$$

$$\tilde{T}^a C^{-1} M_\lambda + C^{-1} M_\lambda T^a = 0 \tag{8}$$

where C is the charge conjugation matrix and $\tilde{}$ denotes transposition. We follow in part the notation of Salam and Strathdee (Salam and Strathdee, 1978). At this stage the Lagrangian involves an arbitrary number of representations (L, M, N) with arbitrary Yukawa couplings h_{LMN} and arbitrary masses M_L, $\bar{M}_L$, M_λ. Restriction on this freedom of choice is achieved by demanding that the theory is free from ultraviolet divergences encountered at the one- and two-loop level. Consider first the coupling g. The one-loop contribution to its renormalization coefficient $b_0^{(1)}$ is given by (Gross and Wilczek, 1973; Politzer, 1973)

$$b_0^{(1)} = -\tfrac{11}{3}C_2(G) + \tfrac{2}{3}T_f(R) + \tfrac{1}{6}T_s(R) \tag{9}$$

where $C_2(G)$ is the quadratic Casimir for the gauge bosons in the adjoint representation G, $T_f(R)$, and $T_s(R)$ are the second indices for two-component fermions and real scalar bosons in representation R of G. The one-loop contribution $(b_0^{(1)})$ of the various $N = 1$, 2, and 4 theories can be worked out by using equation (9). These are given in Table I. The second indices of the fermions $[T_f(R)]$ and the scalar $[T_s(R)]$ in the same representation R are equal and are denoted by $T(R)$. Vanishing of charge renormalization at one loop for the Lagrangian of equation (4) requires (Rajpoot et al., 1983b)

$$3C_2(G) = \sum_R T(R) \tag{10}$$

Table I. One-Loop Contribution to Charge Renormalization in $N = 1, 2$, and 4 Supersymmetric Theories

Supersymmetry	Multiplet and particle content	b_0
$N=1$	Vector $(1^1, \frac{1}{2}^1)$	$3\ C_2(G)$
	Matter $(\frac{1}{2}^1, 0^2)$	$-T(R)$
$N=2$	Vector $(1^1, \frac{1}{2}^2, 0^4)$	$2\ C_2(G)$
	Matter $(\frac{1}{2}^2, 0^4)$	$-2\ T(R)$
$N=4$	Vector $(1^1, \frac{1}{2}^4, 0^6)$	0

This relation fixes *the number and type* of matter multiplets that can be used to construct the one-loop finite theory and was first discussed in reference Rajpoot et al. (1983b). The Yukawa interaction term in the Lagrangian [equations (4) and (5)] does not contribute to the one-loop renormalization of the gauge coupling g. Restriction on it comes from considering the renormalization of g at the two-loop level.

The two-loop contribution (Jones, 1975; Vladimirov and Shirkov, 1979) to the renormalization of the gauge coupling g can be divided into four parts according to the interacting particles, and their contribution is given separately:

(i) Gauge-boson-gauge boson contribution,

$$b_{gg}^{(2)} = -\tfrac{34}{3}C_2(G)^2 \tag{11}$$

(ii) Gauge-boson-two-component fermions in the representation R contribution,

$$b_{gf}^{(2)} = \tfrac{10}{3}C_2(G)T(R) + 2C_2(R)T(R) \tag{12}$$

(iii) Gauge-boson-real-scalars in the representation R,

$$b_{gs}^{(2)} = 2C_2(R)T(R) + \tfrac{1}{3}C_2(G)T(R) \tag{13}$$

(iv) Fermions-scalars contribution coming from the Yukawa coupling (Jones, 1975; Vladimirov and Shirkov, 1979)

$$b_{fs}^{(2)} = \mathrm{Tr}(h\tilde{h}R^aR^b)$$

where h denotes a general Yukawa interaction matrix and $\tilde{h}$ is the conjugate matrix.

Note that the two-loop contributions depend only on the group theoretic factors $C_2(G)$, $T(R)$, $C_2(R)$, and $h_{LMN}h_{L'MN}$ [from the Yukawa interaction term in equation (5)] which are defined as

$$\sum_{b,c} f^{abc}f^{a'bc} = C_2(G)\delta^{aa'} \tag{14}$$

$$\sum_{a,j} R_{ij}^aR_{jk}^a = C_2(R)\delta_{ik} \tag{15}$$

$$\sum_{i,j} R_{ij}^aR_{ji}^b = T(R)\delta^{ab} \tag{16}$$

$$\sum_{M,N} h_{LMN}h_{L'MN} = \Delta_2\delta_{LL'} = (\Delta_2)_{LL'} \tag{17}$$

It is straightforward to work out the two-loop contributions of equations (10) to (13) for the $N=1$ Lagrangian of equation (4) by using its component form as in equation (5). These are given in Table II. The final two-loop contribution is

$$b_0^{(2)} = 4C_2(R)T(R) - 2\Delta_2(R)T(R)g^{-2} \tag{18}$$

where use has been made of the one-loop finiteness condition [equation

Table II. Two-Loop Contribution of $N = 1$ Superfields to the Renormalization of the Gauge Coupling

Multiplet and particle content	Interacting particles	Contribution in units of $(g/4\pi)^4$
Vector $(1^1, \frac{1}{2}^1)$	(gauge boson–gauge boson)	
	+ (gauge boson–gauge fermion)	$-6C_2(G)^2$
Matter $(\frac{1}{2}^1, 0^2)$	(gauge boson–fermion)	
	+ (gauge boson–scalar)	$6C_2(R)T(R)+4C_2(G)T(R)$
	Yukawa:	
	$\lambda^a R^a_{ij}\bar{\psi}^L_i(A+\gamma_5 B)^L_j$	$-2C_2(R)T(R)-2C_2(G)T(R)$
	$h_{LMN}\bar{\psi}^L(A+\gamma_5 B)^M\psi^N$	$-2\Delta_2(R)T(R)/g^2$

(11)] for further simplification. That the gauge coupling be unrenormalized at two loops requires $b_0^{(2)}$ to vanish, i.e.,

$$\Delta_2(R) = g^2 2C_2(R) \tag{19}$$

This condition fixes the magnitude and form of the Yukawa coupling in the Lagrangian of equation (4). Hence finiteness of gauge coupling at one and two loops fixes the representation content and their possible Yukawa interactions.

2.1. Nonrenormalization of the Triple Vertex Coupling

The one- and two-loop nonrenormalization conditions [equations (9) and (19)] for the gauge couplings fix also the form of the triple vertex coupling h_{LMN}. These two finiteness conditions also ensure that h_{LMN} is not renormalized up to two loops. To demonstrate this we make use of the superfield calculations. Dimensional arguments lead to the remarkable result that the triple vertex corrections are finite. Hence only the chiral superfield wave function renormalization need be considered. It is then sufficient to show that the wave function renormalization vanishes at the one- and two-loop level due to the finiteness conditions of equations (9) and (19). In the absence of masses there are only two one-loop diagrams contributing to the chiral superfield wave function renormalization. These are shown in Figure 1. Their contribution is

$$\frac{1}{2}\mu^{\varepsilon}\int \frac{d^4p}{(2\pi)^4}\frac{d^Dk}{(2\pi)^D}\, d^4\theta \frac{\phi(-P,\theta)[\Delta_2 - 2R^aR^ag^2]\phi(P,\theta)}{k^2(p-k)^2} \tag{20}$$

where the loop integrals are regularized in D space-time dimensions, $\varepsilon = (4-D)/2$ and repeated indices are contracted. This contribution vanishes due to the finiteness conditions of equation (19). Following Machasek and Vaugham (1983) the two-loop corrections to the wave function ϕ^L can be worked out as (a) *one-loop corrected* propagator insertions to Figure 2 and

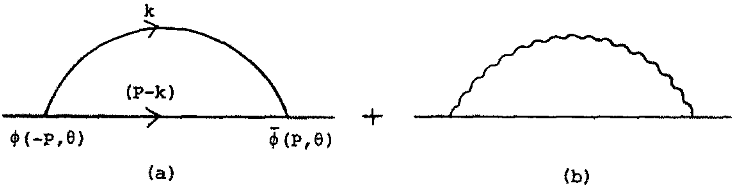


Fig. 1. Diagrams contributing to one-loop chiral superfield wave function renormalization.

a set of additional two-loop diagrams as shown in Figure 3. The one-loop corrected $\bar{\phi}\phi$ vertex is given by equation (20) (which is zero), while the diagrams contributing to the vector propagator are as shown in Figure 2. Their contribution is

$$\mu^{\varepsilon} g^{2} \int \frac{d^{4}p}{(2\pi)^{4}} \frac{d^{D}k}{(2\pi)^{D}} d^{4}\theta V(-P, \theta)\left[\sum_{R} T(R)-3C_{2}(G)\right]$$
$$\times\left[-k^{2}+\frac{1}{2}k^{\alpha\beta}D_{\alpha}(P)\bar{D}_{\beta}(p)+\frac{1}{16}D^{2}(P)\bar{D}^{2}(P)\right]V(P, \theta) \qquad (21)$$

which is zero due to the finiteness condition of equation (9). Hence there are no one-loop propagator insertions in the two-loop ϕ-wave function

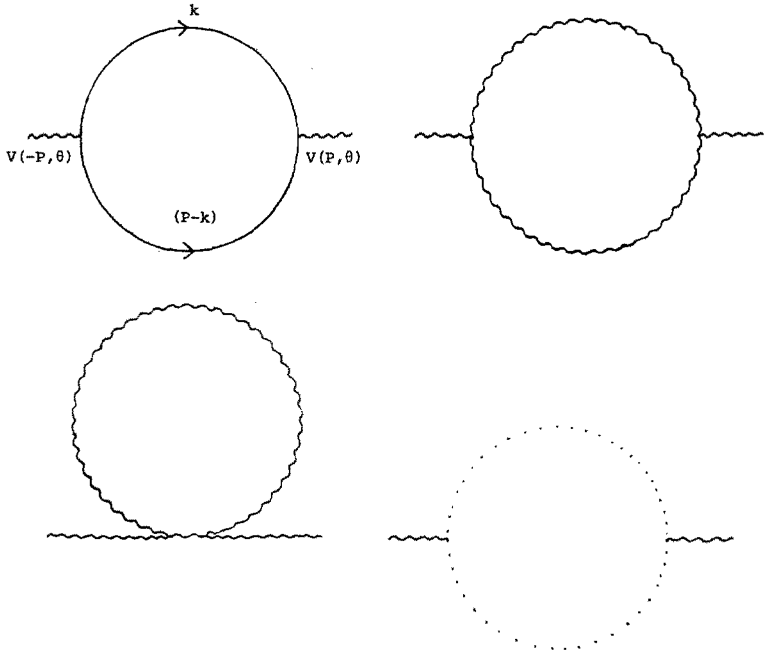


Fig. 2. One-loop corrections to the vector superfield propagator.

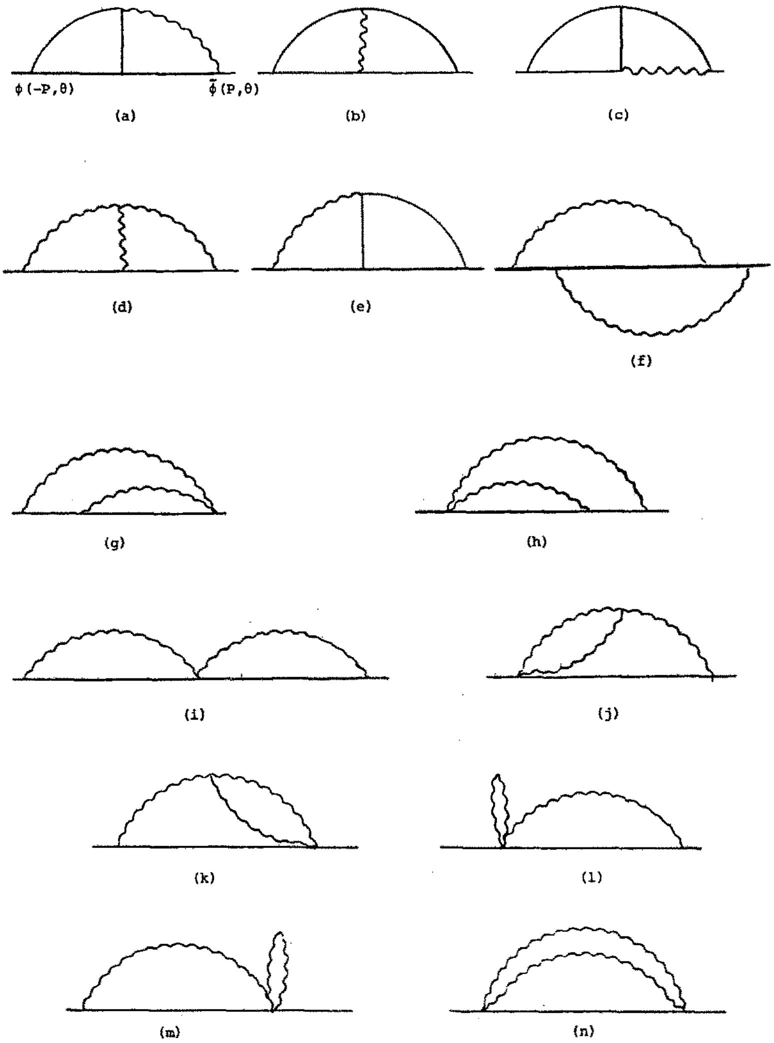


Fig. 3. Diagrams contributing to two-loop chiral superfield wave function renormalization.

renormalization. The remaining two-loop diagrams are shown in Figure 3. Their contribution is

$$g^4\mu^{2\varepsilon}\int\frac{d^4p}{(2\pi)^4}\bar{\phi}(-P,\theta)\Big\{-[(hR^ahR^a)I_2]g^{-2}-[\text{finite contribution}]$$

$$-g^{-2}[(hhR^aR^a)I_2]-\left[\frac{1}{2}C_2(G)C_2(R)\left(\frac{1}{2}I_1-I_2\right)\right]$$

$$
\begin{aligned}
&-\left[\frac{1}{2}\Delta_2 C_2(R) I_2\right]-\left\{\left[C_2^2(R)-\frac{1}{2}C_2(R)C_2(G)\right]I_1\right\}\\
&+\left[C_2^2(R)-\frac{1}{4}C_2(R)C_2(G)\right]\{[I_2]+[I_1]+[I_2]\\
&+[0]+[0]+[0]+[0]+[0]\}\phi(P,\theta)
\end{aligned}
\tag{22}
$$

where I_1 and I_2 are the divergent integrals:

$$
I_1=\int\frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\frac{1}{k^2(k+p)^2l^2(l+p)^2}
\tag{23}
$$

$$
I_2=\int\frac{d^Dk}{(2\pi)^D}\frac{d^Dl}{(2\pi)^D}\frac{1}{k^2l^2(k+p)^2(k+l)^2}
\tag{24}
$$

Shifts of integration variables have been employed in the various integrals arising from the contributions of Figure 3 to bring them in the standard form I_1, I_2 of equations (23), (24). The group factors can be simplified further by using the gauge invariance condition (6). This simplifies equation (22) to

$$
g^4\mu^{2\varepsilon}\int\frac{d^4p}{(2\pi)^4}\phi(-P,\theta)[-\Delta_2 C_2(R)g^{-2}+2C_2^2(R)]\bar{\phi}(P,\theta)I_2
$$

and this vanishes due to the finiteness condition of equation (10). Hence there is no wave function renormalization at two loops and the triple vertex coupling h_{LMN} is finite. We have also checked that there are no counter term contributions arising from one-loop corrections to trilinear couplings.

3. ANOMALY CANCELLATION

Renormalizability of field theories necessitates use of representations that are free from triangle anomalies (Machacek and Vaugham, 1983). These representations are also required to be complex since the quarks and leptons of the three families $\psi^{(e)}$, $\psi^{(\mu)}$, $\psi^{(\tau)}$ lie in complex representations of the particle interaction group $SU(3)\times SU(2)\times U(1)$. Another reason for using complex representations is to avoid the problem of *unnatural fine tuning of particle masses* which in principle could be as large as the natural unification scale of the theory (Adler, 1969; Bell and Jackiw, 1969). The representations of all classical groups except A_M [$\equiv SU(M+1)$] and $O(6)$ are anomaly free. This fact is related to the absence of a third-order Casimir invariant for the Lie algebra of all classical groups except $SU(M)$ and $O(6)$. Since $O(6)$ is locally isomorphic to $SU(4)$, this case is contained in the discussion of anomaly-free complex representations of $SU(M)$. The Young tableaux, dimensionality $d(R_i)$, and the anomaly index $[A(R_i)]$ of some interesting representations of $SU(M)$ are

$$\begin{array}{ccc}
R_i & d(R_i) & A(R_i) \\
\underbrace{\square\square\cdots\square}_{p\text{ boxes}} & \dfrac{(M+P-1)!}{(M-1)!P!} & \dfrac{(M+P)!(M+2P)!}{(M+2)!(P-1)!} \\
\overbrace{\begin{array}{l}\square\square\cdots\square\\ \square\end{array}} & P\dfrac{(M+P-1)!}{(M-1)!(P+1)!} & \dfrac{(M+P-1)!}{(M+2)!P!}F(M,P) \\
p\text{ boxes}\left\{\begin{array}{c}\square\\ \vdots\\ \square\end{array}\right. & \dfrac{M!}{(M-P)!P!} & \dfrac{(M-2P)!(M-3)!}{(M-P-1)!(P-1)!} \\
p\text{ boxes}\left\{\begin{array}{l}\square\square\\ \vdots\\ \square\end{array}\right. & \dfrac{P(M+1)!}{(M-P)!(P+1)!} & \dfrac{(M-3)!}{(M-P)!P!}G(M,P)
\end{array} \tag{25}$$

where

$$\begin{aligned} F(M,P) &= M^3(P-1)+M^2(3P^2-4P-3)+M(2P^3-5P^2-5P-2) \\ &\quad -2P(P+1) \end{aligned} \tag{26}$$

$$G(M,P)=M^3-M^2(3P-4)+M(2P^2-5P-7)+2P^2+4P+2$$

The $A(R_i)$ are normalized such that the anomaly of the fundamental representation is +1. The anomaly indices of other representations can be worked out using the following two formulas iteratively,

$$A(R_i \oplus R_j)=A(R_i)+A(R_j) \tag{27}$$

$$A(R_i \otimes R_j)=\dim(R_i)A(R_j)+\dim(R_j)A(R_i) \tag{28}$$

The condition that the representations used to build up the finite field theory be anomaly free implies

$$\sum_R A(R)=0 \tag{29}$$

The anomaly of the adjoint representation is zero, i.e., $A(\square\square)=0$. This can be seen by using equation (28) and the fact that the anomalies of $\{\square, \begin{smallmatrix}\square\\ \square\end{smallmatrix}\}$ representations are $(-1,0)$. Thus equation (29) is trivially satisfied in the

$N=4$ super Yang-Mills theory since all superfields belong to the adjoint representation of the gauge group G.

In the case of $N=2$ theory (Georgi, 1979; Eichten et al., 1982), the vector superfield consists of one $N=1$ vector superfield and one $N=1$ chiral superfield in the adjoint representation. Hence it is anomaly free. The matter hypermultiplet consists of two chiral $N=1$ superfields, one in representation $\bar{R}$ and the other one in the conjugate representation $\bar{R}$ of the symmetry group G. This construction of the matter hypermultiplet is the consequence of the central charge constraint. Since the representation conjugate to $R=(\lambda_1, \lambda_2, \ldots, \lambda_{M-1})$ is $\bar{R}=(\lambda_{M-1}, \ldots, \lambda_2, \lambda_1)$, their anomalies are related by

$$A(\bar{R})=-A(R) \tag{30}$$

Hence the hypermultiplet of the $N=2$ theory also satisfy the anomaly constraint of equation (29). That the anomalies of different representations must be required to cancel among themselves is a new feature of finite $N=1$ field theories. The kind of representations that are admissible in these theories is also restricted by the constraint imposed by the vanishing of the one-loop charge renormalization (Capper and Rajpoot, 1984). This constraint is discussed for all classical groups in the following section. Also in order to discuss the vanishing of charge renormalization we need the quadratic Casimir invariants of all classical groups. These are listed in Table III for convenience.

4. REPRESENTATIONS OF CLASSICAL GROUPS SATISFYING THE CONSTRAINTS OF NO CHARGE RENORMALIZATION AND ANOMALIES CANCELLATION

4.1. $A_{M-1} \equiv SU(M)$

The anomaly constraint is only relevant to this family of groups. In Table IV, the Young tableaux, the dimensionality $d(R_i)$, the second index $T(R_i)$, and the anomaly index $A(R_i)$ of all representations satisfying the constraint

$$T(R) \leq 3C_2(G)$$

are given.

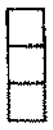
The last representation is only relevant for $M=4$. For $M=3$, it is the $\bar{6}$ of $SU(3)$. For finite grand unification, we require groups with rank greater or equal to four ($M \geqslant 5$). However it is conceivable that quark and leptons

Table III. The Dimension and $C_2(G)$ of the Adjoint Representations of all Classical Groups

Group G	Dimension	$C_2(G)$
$SU(M)=A_{M-1}$	M^2-1	M
$SO(2M+1)=B_M$	$M(2M+1)$	$2M-1$
$Sp(2M)=C_M$	$M(2M+1)$	$2M+2$
$SO(2M)=D_M$	$M(2M-1)$	$2M-2$
E_6	78	12
E_7	133	18
E_8	248	30
F_4	52	9
G_2	14	112

are composite. In this case groups of rank less than four (i.e., $M \leq 5$) are relevant for constructing finite preon models. Table IV covers both possibilities. Here we consider finite grand unification. Each family is understood to constitute a 5 and a $\overline{10}$ of $SU(5)$. The particle spectrum of

Table IV. Dimensionability, Anomaly index, and the Second Indices of $SU(M)$ Representations Allowed by the Constraint of One-Loop Finite Charge Renormalization

Representation	Dimension	$A(R_i)$	$T(R_i)$	Range of N
	M	1	$\frac{1}{2}$	$M \geq 2$
	$\frac{M(M-1)}{2}$	$M-4$	$\frac{M-2}{2}$	$M \geq 4$
	$\frac{M(M+1)}{2}$	$N+4$	$\frac{M+4}{2}$	$M \geq 3$
	$\frac{M(M-1)(M-2)}{3}$	$\frac{(M-3)(M-6)}{2}$	$\frac{(M-2)(M-3)}{4}$	$6 \leq M \leq 17$
$(M-1)$ boxes 	(M^2-1)	0	M	$M \geq 2$
	$\frac{M}{3}(M^2-1)$	(M^2-9)	$\frac{M^2-3}{2}$	$M \leq 6$
	$\frac{M^2}{12}(M^2-1)$	$\frac{M(M^2-16)}{3}$	$\frac{M(M^2-4)}{6}$	$M \leq 4$

the finite $SU(5)$ theory will consist of a fixed number of 5's, $\bar{5}$'s, 10's, $\overline{10}$'s, and a 24. Out of this, at least three 5's and three $\overline{10}$'s should survive. The rest should consist of equal numbers of $(5, \bar{5})$'s and $(10, \overline{10})$'s which pair off to acquire superheavy mass according to the survival hypothesis of Georgi (Georgi, 1979; Eichten et al., 1982). The combination in the proceedings is always anomaly free.

How many light families can an $N=1$ finite model admit? Let there be $(f+a)$ number of 5's, a number of $\bar{5}$'s, $(f+b)$ number of $\overline{10}$'s and b number of 10's. Note that a number of the 5, and $\bar{5}$'s annihilate to get superheavy. Similarly for the b number of 10 and $\overline{10}$'s. This leaves f $(5+\overline{10})$'s and hence f families. We also require the adjoint representation of $SU(5)$ for the descent of $SU(5)$ to $SU(3)\times SU(2)\times U(1)$. The anomaly constraint is trivially satisfied,

$$(f+a)A(5)+aA(\bar{5})+(f+b)A(\overline{10})+bA(10)+A(24)=0 \qquad (31)$$

Vanishing of charge renormalization [equation (10)] gives

$$\tfrac{1}{2}(f+2a)+\tfrac{3}{2}(f+2b)+5=15 \qquad (32)$$

Table V gives all the solutions of equation (32) that contain three or more families. Note that the minimal finite $SU(5)$ model can contain five families at most. It is interesting to note that this constraint on the number of families in the $SU(5)$ finite field theories is not far from the one deduced from cosmological considerations (Olive et al., 1981; Turner, 1980) $(f\leq 4)$. For a more complete discussion on $SU(M)$ groups and anomaly cancellation see Rajpoot and Taylor (1984).

4.2. $SO(2M+1)=B_M$

The representations of this chain of groups with $T(R)<3C_2(G)$ are given in Table VI. The representations of high dimensionality are relevant

Table V. Particle Content of $N=1$ Finite $SU(5)$ Models

f	$\bar{5}$	5	$\overline{10}$	10	24
3	4	7	3	0	1
	4	1	4	1	1
4	2	5	3	0	1
5	0	5	5	0	1

Table VI. The Dimension and Dynkin Index of $SO(2M+1)$ [$\equiv B_M$] Representations

Representation	Dimension	$T(R)$	Range of M
□	$2M+1$	1	$M \geq 2$
Spinor	2^M	2^{M-3}	$2 \leq M \leq 7$
□ □ (Adjoint)	$M(2M+1)$	$2M-1$	$M \geq 2$
□□	$(M+1)(2M+1)$	$2M+3$	$M \geq 2$
□ □ □	$\frac{M}{3}(4M^2-1)$	$M(2M-3)$	$2 \leq M \leq 3$

for spontaneous symmetry breaking. Finiteness as required by equation (10) implies

$$n_1 + n_2 2^{M-3} + n_3(2M-1) + n_4(2M+3) + n_5 M(2M-3) = 3(2M-1) \quad (33)$$

where n_i's are the multiplicities of the representations as in Table VI and $M \geq 2$. Some solutions of equation (33) are the following:

(i) $n_1 = n_2 = n_4 = n_5 = 0$, $n_3 = 3$. This theory is the $N = 4$ super Yang-Mills theory and hence finite to all orders.

(ii) $n_2 = n_4 = n_5 = 0$. Equation (33) reduces to $n_1 + (n_2 - 3)(2M - 1) = 0$. $0 \leq n_3 < 3$. Grand unified theories built with these constraints can contain a large number of conventional families accompanied by their mirror counterparts. The neutrinos identifying the families in excess of four to five are required to be massive (>2 GeV) to avoid conflict with helium abundance in the universe (Olive et al., 1981; Turner, 1980).

(iii) $n_1 = n_4 = n_5 = 0$. Only the spinor representation and the adjoint representations are active. These representations are required to satisfy

$$n_2 2^{M-3} + (n_3 - 3)(2M - 1) = 0 \quad (34)$$

The solutions $[n_2, n_3]$ of this equation are

$$O(5): \quad (9, 0), (6, 1), (3, 2)$$

$$O(7): \quad (15, 0), (10, 1), (5, 2)$$

$$O(9): \quad (7, 1)$$

There are no other solutions to equation (34).

(iv) A mixture of spinors, fundamentals, and adjoint representations $n_4 = n_5 = 0$. The equation to be satisfied is $n_1 + n_2 2^{M-3} + (n_3 - 3)(2M - 1) = 0$.

The solutions are

$$\begin{array}{lll}
SO(5); & (n_1, n_2, n_3) = (n, 2(9-n), 0), & n_1 \leq 9 \\
& \qquad\qquad\;\; = (n, 2(6-n), 1), & n_1 \leq 6 \\
& \qquad\qquad\;\; = (n, 2(3-n), 2), & n_1 \leq 3 \\
SO(7); & (n_1, n_2, n_3) = (n, 15-n, 0), & n \leq 15 \\
& \qquad\qquad\quad\;\; (n, 10-n, 1), & n \leq 10 \\
& \qquad\qquad\quad\;\; (n, 5-n, 2), & n \leq 5 \\
SO(9); & (n_1, n_2, n_3) = (21-2n, n, 0), & n \leq 10 \\
& \qquad\qquad\quad\;\; (14-2n, n, 1), & n \leq 7 \\
& \qquad\qquad\quad\;\; (7-2n, n, 2), & n \leq 3 \\
SO(11); & (n_1, n_2, n_3) = (27-4n, n, 0), & n \leq 6 \\
& \qquad\qquad\quad\;\; (18-4n, n, 1), & n \leq 4 \\
& \qquad\qquad\quad\;\; (9-4n, n, 2), & n \leq 2 \\
SO(13); & (n_1, n_2, n_3) = (33-8n, n, 0), & n \leq 4 \\
& \qquad\qquad\quad\;\; (22-8n, n, 1), & n \leq 2 \\
& \qquad\qquad\quad\;\; (11, 0, 2), (3, 1, 2) & \\
SO(15); & (n_1, n_2, n_3) = (23, 1, 0), (6, 2, 0) & \\
& \qquad\qquad\quad\;\; (10, 1, 1) & \\
SO(17); & (n_1, n_2, n_3) = (13, 1, 0) &
\end{array}$$

There are no solutions beyond $M > 8$.

4.3. $SP(2M) = C_M$

This chain of groups, although much used in classical mechanics, has not been favored for describing schemes of unification (Rajpoot, 1981). This chain will therefore be considered briefly. Since $SP(2) \simeq SU(2)$, $SP(4) \simeq SO(5)$, we require $M > 3$ to avoid overlapping discussion in A_{M-1} and B_M. All representations are anomaly free. Table VII gives the dimension

Table VII. The Dimension and Second Index $T(R)$ of $SP(2M)$ Representations (R)

Representation	Dimension	$T(R)$	Range of M
□	$2M$	1	$M \geq 3$
□ □	$M(2M-1)-1$	$2M-2$	$M \geq 3$
□□ = adjoint	$M(2M+1)$	$2M+2$	$M \geq 3$

and Dynkin index of $SP(2M)$ representations. Finiteness as required by equation (10) implies

$$n_1+(2M-2)n_2+(2M+2)n_3=3(2M+2) \tag{35}$$

The solutions of this equation are as follows:

(a) $n_1=n_2=0$, $n_3=3$. This theory coincides with the $N=4$ super Yang-Mills theory.

(b) $n_2=0$, $n_1=(3-n_3)(2M+2), 0\leqslant n_3<3$. The maximal subgroup of $Sp(2M)$ is $SU(M)\times U(1)$. $SU(16)$ can accommodate one family of conventional fermions. Hence $SP(32)$ can serve as a candidate for $N=1$ finite unification. The number of fermion families is $34(3-n_3)$. These are accompanied by equal numbers of mirror families.

(c) $n_1, n_2, n_3\neq 0$. The general solutions of equation (35) are

$$(n_1, n_2, n_3)=(2M+6, 1, 1), (8, 2, 1)$$
$$(10-2M, 3, 1) \text{ for } 3\leq M\leq 5, (4, 1, 2)$$

Many schemes of unification are allowed for large enough M. The choice of M [i.e., restriction on $Sp(2M)$] is only restricted by the solution $(10-2M, 3, 1)$.

4.4. $SO(2M)=D_M$

The representations that have second index $T(R_i)$ less than or equal to $3C_2(G)$ are tabulated in Table VIII. Only the fundamental spinor, second-, and third-rank antisymmetric tensor representations are allowed. The third-rank antisymmetric representation is allowed for the $SO(8)$ group since the groups $\{SO(4), SO(6)\}$ are isomorphic to $SU(2)\times SU(2)$ and $SU(4)$ which have already been dealt with under the A_M chain of groups. If the multiplicities of the first three representations are denoted by n_i, then equation (10) gives

$$n_1+2^{M-4}n_2+(2M-2)n_3=3(2M-2) \tag{36}$$

Table VIII. The Dimension and Second Index $T(R)$ of $SO(2M)$ Representations

Representation	Dimensionality	$T(R)$	Range of M
Fundamental	$2M$	1	$M\geq 4$
Spinor	2^{M-1}	2^{M-4}	$5\leq M\leq 9$
Second-rank antisymmetric ≡ adjoint	$M(2M-1)$	$2M-2$	$M>3$
Third-rank antisymmetric	$\frac{M}{3}(2M-1)(2M-2)$	$(2M-1)(M-2)$	Only $M=4$ allowed

The allowed solutions are as follows:

(i) $n_1 = n_2 = 0$, $n_3 = 3$. The theory becomes the explicitly broken $N = 4$ super Yang–Mills theory.

(ii) $n_1 = n + \bar{n}$, $n_2 = m + \bar{m}$, $n_3 = 1$. The theory becomes the $N = 2$ theory with n fundamentals and m spinors.

(iii) $SO(8)$. This admits the third-rank antisymmetric representation in Table VIII. Finiteness requires $n_1 + n_2 + 6n_3 + 14n_4 = 18$. The solutions are

$$(n_1, n_2, n_3, n_4) = (18 - n, n, 0, 0), (12 - n, n, 1, 0),$$

$$(6 - n, n; 2, 0), (4 - n, n, 0, 1)$$

$$SO(10); \quad (n_1, n_2, n_3) = (24 - 2n, n, 0), (16 - 2n, n, 1), (8 - 2n, n, 2)$$

$$SO(12); \quad (n_1, n_2, n_3) = (30 - 4n, n, 0), (20 - 4n, n, 1), (10 - 4n, n, 2)$$

$$SO(14); \quad (n_1, n_2, n_3) = (36 - 8n, n, 0)(24 - 8n, n, 1)(12 - 8n, n, 2)$$

$$SO(16); \quad (n_1, n_2, n_3) = (42, 0, 0), (26, 1, 0), (10, 2, 0)$$

$$(28, 0, 1), (12, 1, 1), (14, 0, 2)$$

$$SO(18); \quad (n_1, n_2, n_3) = (48, 0, 0), (16, 1, 0), (32, 0, 1)$$

$$(0, 1, 1), (16, 0, 2),$$

$$SO(20); \quad (n_1, n_2, n_3) = (54, 0, 0), (36, 0, 1), (18, 0, 2)$$

In the above $n_1, n_2, n_3 \geq 0$ and the spinor representations of $SO(10)$, $SO(14)$, $SO(18)$ are complex. A consequence of the restriction on the allowed representations is that only certain intermediate stages of symmetry breaking are allowed. Consider the following descent of $SO(10)$ (Rajpoot, 1980):

$$SO(10) \xrightarrow{54} SU(2) \times SU(2) \times SU(4) \tag{37}$$

The traceless symmetric 54 has second index $T(54) = 165/2$ and hence is not admissible. Therefore $SO(10)$ cannot break to the Pati–Salam symmetry group (Pati and Salam, 1974) $SU(2)_L \times SU(2)_R \times SU(4)$. However, the descent

$$SO(10) \xrightarrow{45} SU(4) \times SU(2) \times U(1) \xrightarrow{45} SU(3) \times SU(2) \times U(1) \times U(1)$$

$$\xrightarrow{16} SU(3) \times SU(2) \times U(1) \xrightarrow{16} SU(3) \times U(1)_{\mathrm{em}} \tag{38}$$

is allowed as it requires 2 $\underline{45}$'s and 2 $\underline{16}$'s. This combination satisfies exactly the finiteness equation (36). The physical consequence of this interesting descent is that the charged right-handed gauge bosons are very massive

Table IX. Representations Allowed by Finiteness Condition in the Case of Exceptional Groups

Group	Fundamental	Adjoint
E_6, G_2	12	0
	8	1
	4	2
	0	3
E_7, F_4	9	0
	6	1
	3	2
	0	3
E_8	—	3

$\simeq 10^{10}$ GeV for $\sin^2 \theta_w \simeq 0.22$, $\alpha_s = 0.10$. The second neutral gauge boson can be as light as 200 GeV (Rajpoot, 1982).

The chain of exceptional groups can accommodate at most two low-dimensional representations since the second index, $T(R_i)$ of other representations exceeds $3C_2(G)$. The solutions of the finiteness condition as expressed by equation (10) are given in Table IX. In all cases when three adjoint representations are used the theory coincides with the $N=4$ super Yang–Mills theory.

5. MASS INSERTIONS

The spurion mass terms introduced in the Lagrangian of equation (4) break the $N=1$ supersymmetry. These spurion mass insertions are soft in the sense that their presence induces only logarithmic infinities. In the absence of the spurion insertions [terms proportional to θ^2 and $\theta^2\bar{\theta}^2$ in equation (4)] the Lagrangian exhibits $N=1$ supersymmetry. Elimination of the auxiliary fields has two effects:

(a) Mass-dependent cubic interaction terms appear.

(b) The masses at the tree-level satisfy the supertrace sum rule

$$\sum_{s=0,1/2} (-1)^{2s+1}(2s+1)M_s = 0$$

This can be checked by inspecting the component form of the Lagrangian in equation (5) with $M_\lambda = \hat{0} = \bar{M}_L$. The mass-dependent cubic interactions are crucial to cancel infinities between fermion loops and scalar boson loops. In analogy with this situation the addition of mass terms for the gauginos via the spurion $M_\lambda \theta^2$ requires the introduction of two further terms, viz. $M_\lambda \theta^2 h_{LMN} \phi^L \phi^M \phi^N$ and $M_\lambda^2 \theta^2 \bar{\theta}^2 \phi^L \phi_L$. It is easy to check that the mass-dependent logarithmic infinity induced in the $\phi\phi$ propagator at the one-loop level cancels if

$$\sum_{M,N} h_{LMN} h^{L'MN} = \Delta_2 \, \delta_{LL'} = 2C_2(R) \, \delta_{LL'} g^2$$

This condition is the same as the one deduced from the vanishing of the renormalization of the gauge couplings at the one- and two-loop levels [equations (10) and (19)]. The spurion $\bar{M}\theta^2$ does not require cubic insertions as the induced infinities cancel among themselves due to tachyonic mass terms for the pseudoscalars. All masses in the Lagrangian of equation (4) satisfy the mass squared sum rule of equation (1). This sum rule guarantees that the theory has no quadratic divergences (Capper and Rajpoot, 1984). This result emerges from considering the effective potential at the one-loop level and extends over to the gauge boson masses after spontaneous symmetry breaking [e.g., (2)]. Since supersymmetry is broken, the vacuum energy is divergent. This divergence can be canceled at the expense of introducing a quartic mass sum rule. However the validity of this sum rule is obscure since the vacuum energy, a possible cosmological constant, is only relevant in the presence of gravity and this has been ignored at the present level of consideration.

6. CONCLUSIONS AND FUTURE OUTLOOK

A systematic study of the explicitly broken $N=4$ super Yang-Mills theory revealed that it has the following properties for finiteness:

(i) It has a universal coupling for gauge and matter interactions.

(ii) The representations are anomaly free.

(iii) The charge renormalization vanishes, i.e., $\beta(g)=0$.

(iv) Masses introduced into the theory are required to satisfy the supertrace sum rule $\sum (-1)^{2S+1}(2S+1)M_S = 0$ at the tree level.

The $N=2$ theory can also be made finite by applying the above criteria. It satisfies (i) and (ii) due to $N=2$ supersymmetry and renormalizability. In the absence of mass scales, finiteness requires (iii) and the representations are limited by the condition $2C_2(G) = T(H_i) = 2T(R_i)$, where $T(H_i) =$ the second index of the hypermultiplet. The masses are required to satisfy the tree-level sum rule of (iv).

The above two examples led us to conjecture that it was possible to build ultraviolet finite field theories out of $N=1$ superfields if the four criteria listed above were satisfied. In the present paper it has been verified that the charge renormalization vanishes at two loops if it vanishes at one loop and the masses require the supertrace sum rule if mass renormalization vanishes at one loop.

Finite $N=4$ and $N=2$ theories which satisfy the above four criteria for finiteness strongly suggest that $N=1$ theories built out of one vector and several matter multiplets are also finite to all loop orders if the above criteria are satisfied. Here we present a heuristic argument that suggests why our conjecture may be true. We consider massless versions of the theory for simplicity.

The $N=4$ theory, which is finite in four dimensions, was constructed by formulating the theory in ten dimensions with $N=1$ supersymmetry. It has only the vector multiplet which must be in the adjoint representation of the gauge group. The reduction to four dimensions splits matter into one vector and three matter ($N=1$) multiplets again all in the adjoint representation. Similarly the Yang–Mills part of the $N=2$ theory comes from the formulation of the theory in six dimensions with $N=1$ supersymmetry. Its reduction to four dimensions consists of one vector and one matter multiplet in the adjoint representation of G. The requirement that $\beta(g)=0$ then fixes the number of allowed matter representations. The central charge constraint requires these matter multiplets to occur in pairs of conjugate representations R and $\bar{R}$ of G. When the representations R are adjoint the $N=2$ theory coincides with the $N=4$ theory. It is natural to expect that this hierarchy in finite field theories should continue. The only change from $N=4$ theory to $N=2$ theory is the representation content. All representations should be admissible provided they are anomaly free (the anomaly of adjoint representation is zero in the case of $N=4$ theory, the anomaly of R consists the anomaly of R in the hypermultiplet of the $N=2$ theory). The only missing member in this hierarchy of finite field theories is the one consisting of one $N=1$ vector multiplet and several $N=1$ matter multiplets. The allowed representations are required to satisfy equation (10) and be free from anomalies.

The present work entertains the existence of the missing member and verifies that there is some truth in this conjecture at least up to two loops in the absence of mass scales. An intriguing feature of perturbative charge renormalization is that the two-loop contribution is determined in terms of the one-loop contribution

$$b_0^{(2)}=2C_2(G)\cdot b_0^{(1)}$$

This result follows from using the result of equation (20) in the expression for the two-loop contribution $b_0^{(2)}$. Since $b_0^{(1)}=0$, then $b_0^{(2)}=0$. Whether this factorization persists to all order is an open question. Work along these lines is in progress.

Given these criteria, we have presented a set of representations of various unifying groups which are finite and can accommodate three families. Which of these are most appropriate requires further study.

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